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Rational Chebyshev Approximation to Certain Entire Functions of Zero Order on the Positive Real Axis

A. R. REDDY

Department of Mathematics, Yeshiva University, New York, N.Y. 10033

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Let $f(z) = \sum_{k=0} a_k z^k$ be an entire function. As usual, write

$$M(r) = \max_{|z|=r} |f(z)| .$$

Let us denote

$$\lambda_{0,n} = \lambda_{0,n}\left(\frac{1}{f}\right) = \inf_{P_n \in \pi_n} \left| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right|_{\mathscr{L}_\infty[0,\infty)} \tag{1}$$

where π_n is the class of all algebraic polynomials of degree at most n.

Recently, we have proved the following result [4, Theorem 7′].

THEOREM 1. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, be an entire function satisfying the assumptions*

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \Lambda + 1 \qquad (0 < \Lambda < \infty) \tag{2}$$

and

$$\limsup_{r\to\infty} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = B_l , \qquad \liminf_{r\to\infty} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = b_l ; \qquad (0 < b_l \leqslant B_l < \infty). \tag{3}$$

Then

$$\limsup_{n\to\infty} \lambda_{0,n}^{n^{-(\Lambda+1)/\Lambda}} < 1. \tag{4}$$

Remark. Theorem 7 of [2] follows from (4).

We prove here the following

THEOREM 2. *Under the assumptions of Theorem* 1,

$$\limsup_{n\to\infty} \lambda_{0,n}^{n^{-(\Lambda+1)/\Lambda}} \geqslant \exp\left(\frac{-\Lambda}{(\Lambda+1)[B_l(\Lambda+1)]^{1/\Lambda}}\right) . \tag{5}$$

For proof we need:

LEMMA 1.[3]. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be an entire function satisfying the assumptions* $0 < \Lambda < \infty$ *and* $0 < B_l < \infty$. *Then*

$$\limsup_{n\to\infty} \frac{n^{\Lambda+1}}{\{\log | 1/a_n |\}^{\Lambda}} = \frac{(\Lambda + 1)^{\Lambda+1} B_l}{\Lambda^{\Lambda}}. \tag{6}$$

Proof of Theorem 2. Let $\epsilon > 0$. The coefficients of $f(x)$ being non-negative, we have from (3), for $r \geqslant r_0(\epsilon)$ and $0 \leqslant x \leqslant r$,

$$0 < f(x) \leqslant f(r) = M(r) \leqslant \exp(B_l(1 + \epsilon)(\log r)^{\Lambda+1}). \tag{7}$$

From (7), for

$$r = \exp\left(\left[\frac{2n}{B_l(1+\epsilon)}\right]^{1/(\Lambda+1)}\right), \qquad n \geqslant n_0(\epsilon), \tag{8}$$

we obtain

$$0 < f(x) \leqslant f(r) \leqslant e^{2n}. \tag{9}$$

It is clear from (4) that there is a $q > 1$ for which, for all $n \geqslant n_1(\epsilon)$,

$$q^{n^{(\Lambda+1)/\Lambda}} \leqslant \lambda_{0,n}^{-1}. \tag{10}$$

From (9) and (10) we get, for all $n \geqslant \bar{n} \geqslant \max(n_1, n_0)$,

$$0 < f(x) \leqslant f(r) \leqslant e^{2n} < e^{(\log q)\, n^{(\Lambda+1)/\Lambda}} \leqslant \lambda_{0,n}^{-1}. \tag{11}$$

Next, we pick $P_n^* \in \pi_n$, which gives best approximation in the sense of (1). Then from (1),

$$\frac{-f^2(x)}{f(x) + \lambda_{0,n}^{-1}} \leqslant P_n^* - f(x) \leqslant \frac{f^2(x)}{\lambda_{0,n}^{-1} - f(x)}, \qquad 0 \leqslant x \leqslant r. \tag{12}$$

It is easy to derive from (12),

$$\| P_n^* - f(x)\| \leqslant \frac{e^{4n}}{\lambda_{0,n}^{-1} - e^{2n}}, \qquad n \geqslant \bar{n}. \tag{13}$$

Next, let

$$E_n(f) \equiv \inf_{P_n \in \pi_n} \| P_n(x) - f(x)\|_{[0,r]}. \tag{14}$$

We obtain from (13) and (14)

$$E_n(f) \leqslant \frac{e^{4n}}{\lambda_{0,n}^{-1} - e^{2n}} \qquad \text{for all } n \geqslant \bar{n}. \tag{15}$$

By applying a result of Bernstein [1, p. 10] to (14), we obtain

$$E_n \geqslant \frac{a_{n+1} r^{n+1}}{2^{2n+1}}. \tag{16}$$

From Lemma 1 we get for a sequence of values $n = n_p$, $p \geqslant p_0(\epsilon)$, $c = (1-\epsilon)'$.

$$a_{n+1} \geqslant \exp\left(\frac{-(n+1)^{(\Lambda+1)/\Lambda} \Lambda}{(\Lambda+1)^{(\Lambda+1)/\Lambda} B_l^{1/\Lambda} c}\right). \tag{17}$$

From (8), (15), (16), and (17), we get for all such n,

$$\left[\exp\left(\left(\frac{2n}{B_l(1+\epsilon)}\right)^{1/(\Lambda+1)} - \frac{(n+1)^{(\Lambda+1)/\Lambda} \Lambda}{(\Lambda+1)^{(\Lambda+1)/\Lambda} B_l^{1/\Lambda} c}\right)\right] 2^{-(2n+1)} \leqslant \frac{e^{4n}}{\lambda_{0,n}^{-1} - e^{2n}}. \tag{18}$$

A simple calculation based on (18), gives for such n,

$$\lambda_{0,n}^{-1} \leqslant 2^{8n} \exp\left(\frac{(n+1)^{(\Lambda+1)/\Lambda} \Lambda}{(\Lambda+1)^{(\Lambda+1)/\Lambda} B_l^{1/\Lambda} c}\right). \tag{19}$$

From (19) we easily obtain the required result by noting that $c \to 1$.

References

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